On the summability of formal solutions for singular first-order linear partial differential equations

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Abstract

This article is concerned with the study of the Borel summability of divergent power series solutions for singular first-order linear partial differential equations of nilpotent type. We introduce three main results obtained by the author. In order to assure the Borel summability of divergent solutions, global analytic continuation properties for coefficients are required despite the fact that the domain of the Borel sum is local.

1. Introduction

In this paper we study the following first-order linear partial differential equation with two complex variables:

$$\{A(x,y)D_x + B(x,y)D_y + 1\}u(x,y) = f(x,y), \quad (1)$$

where $x,\, y \in \mathbb{C}$, $D_x = \partial/\partial x$, $D_y = \partial/\partial y$. The coefficients A, B and f are holomorphic at $(x,y) = (0,0) \in \mathbb{C}^2$.

Throughout this paper we always assume the following three fundamental conditions:

$$A(x,0) \equiv 0, \quad (2)$$

$$\frac{\partial A}{\partial y}(0,0) \neq 0, \quad (3)$$

$$B(x,0) \equiv \frac{\partial B}{\partial y}(x,0) \equiv 0. \quad (4)$$

Remark 1 Conditions (2) and (4) imply $A(0,0) = B(0,0) = 0$, which means that (1) is *singular at the origin*. Moreover, it follows from (2)–(4) that the Jacobi matrix $\partial(A,B)/\partial(x,y)|_{(x,y)=(0,0)}$ is a nilpotent matrix

$$\begin{pmatrix} 0 & (\partial A/\partial y)(0,0) \\ 0 & 0 \end{pmatrix}.$$

In this sense, our equation is called of *nilpotent type*.

First of all, let us consider the existence of formal power series solutions $u(x,y) = \sum_{m,n=0}^{\infty} u_{mn}x^m y^n$ around $(x,y) = (0,0)$. Then, under the above conditions we can prove the unique existence of $u(x,y)$. Moreover, we see that it takes the form of $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n$, where $u_n(x)$ are holomorphic in a common neighborhood of $x = 0$. However, because of the singularity of (1) at the origin, this formal power series solution $u(x,y)$ with respect to y-variable diverges in general and the rate of divergence is characterized in terms of the Gevrey index (cf. Definition 1, (i)–(iii) and Theorem 1). So, we are interested in the Borel summability of such a divergent solution (cf. Definition 1, (iv)–(vi)). *Our main purpose is to obtain the conditions under which the divergent solution is Borel summable.*

In the theory of ordinary differential equations, there are many studies concerning the Borel summability for divergent power series solutions, and we can see many significant results in Balser's books [1] and [3]. On the other hand, in the theory of partial differential equations, such studies started recently. The first contribution is rendered by Lutz-Miyake-Schäfke[11], where complex heat equations are dealt with. Balser[2, 4], Balser-Miyake[5] and Miyake[12] generalized the result in [11]. In Ōuchi[13] also, we can find some interesting results for greatly general linear partial differential equations. We remark that our equation (1) is a different type of equation from theirs, and that in the above articles we can see quite different phenomena from ours.

Now, the content of this paper is as follows. In Chapter 2, we give the definition of divergent power series of the Gevrey type and the Borel summability. Moreover, we state the theorem which assures the

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unique existence of divergent power series solutions (Theorem 1). From Chapter 3, we consider the problem of the Borel summability. In Chapter 3. we place some restrictions on equations, and according to those restrictions we divide equations into three classes. In Chapter 4, for each class, we give conditions which the coefficients should satisfy in order to assure the Borel summability of the divergent solutions. These conditions were obtained in [8, 9, 10]. *Global analytic continuation properties for coefficients are required.*

2. Definition and Fundamental Result

Definition 1 (i) $\mathcal{O}[R]$ denotes the ring of holomorphic functions on the closed ball $B(R) = \{x \in \mathbb{C};\ |x| \le R\}$, where R is a positive number.

(ii) The ring of formal power series in y ($\in \mathbb{C}$) over the ring $\mathcal{O}[R]$ is denoted as $\mathcal{O}[R][[y]]$:

$$\mathcal{O}[R][[y]] = \left\{ u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n;\ u_n(x) \in \mathcal{O}[R] \right\}. \tag{5}$$

(iii) We say that $u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n$ ($\in \mathcal{O}[R][[y]]$) *belongs to* $\mathcal{O}[R][[y]]_2$, if there exist some positive constants C and K such that

$$\max_{|x|\le R} |u_n(x)| \le C K^n n! \tag{6}$$

for all $n = 0, 1, 2, \ldots$. The suffix 2 of $\mathcal{O}[R][[y]]_2$ expresses the Gevrey index of power series. Elements of $\mathcal{O}[R][[y]]_2$ are divergent series in general.

(iv) For $\theta \in \mathbb{R}$, $\kappa > 0$ and $0 < \rho \le +\infty$, the sector $S(\theta, \kappa, \rho)$ in the universal covering space of $\mathbb{C} \setminus \{0\}$ is defined by

$$S(\theta,\kappa,\rho) = \left\{ y;\ |\arg(y) - \theta| < \frac{\kappa}{2},\ 0 < |y| < \rho \right\}. \tag{7}$$

We refer to θ, κ and ρ as the *bisecting direction*, the *opening angle* and the *radius* of $S(\theta,\kappa,\rho)$, respectively.

(v) Let $u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$ and let $U(x,y)$ be a holomorphic function on $X = B(R) \times S(\theta,\kappa,\rho)$. Then we say that $U(x,y)$ *has* $u(x,y)$ *as an asymptotic expansion of the Gevrey order* 2 *in* X if the following asymptotic extimates hold: there exist some positive constants C and K such that

$$\max_{|x|\le R} \left| U(x,y) - \sum_{n=0}^{N-1} u_n(x) y^n \right| \le C K^N N! |y|^N \tag{8}$$

for all $y \in S(\theta,\kappa,\rho)$ and $N = 1, 2, \ldots$. Then we write this as

$$U(x,y) \cong_2 u(x,y) \ \text{ in } \ X.$$

(vi) Let $u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$. We say that $u(x,y)$ *is Borel summable in a direction* θ if there exists a holomorphoic function $U(x,y)$ on $X = B(r) \times S(\theta,\kappa,\rho)$ for some $0 < r \le R$ and $\kappa > \pi$ which satisfies $U(x,y) \cong_2 u(x,y)$ in X. A given divergent power series $u(x,y) \in \mathcal{O}[R][[y]]_2$ is not necessarily Borel summable in general. However, if $u(x,y)$ is Borel summable in a direction θ, then we see that the above holomorphic function $U(x,y)$ is unique (cf. Balser[1, 3]). So we call this unique $U(x,y)$ *the Borel sum of* $u(x,y)$ *in a direction* θ.

Now we already know the following fact, which will be fundamental in the argument below.

Theorem 1 *Let us assume* (2)–(4). *Then the equation* (1) *has a unique formal power series solution* $u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$ *for some* $R > 0$.

On the basis of Theorem 1, we can study the coming problem: the Borel summability of the formal solution.

3. Rewriting of Equation

Let us divide the problem into the following two parts:

(i) *When is the formal solution* $u(x,y)$ *Borel summable in a given direction* θ*?*

(ii) *Is the Borel sum* $U(x,y)$ *a solution?*

By the following theorem, problem (ii) is always sovled affirmatively.

Theorem 2 (cf. [7]) *Let* $u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$ *be the formal solution of* (1), *and let us assume that* $u(x,y)$ *is Borel summable in some direction* θ. *Then its Borel sum* $U(x,y)$ *is a holomorphic solution of* (1).

Hereafter, we consider problem (i). To begin with, let us rewrite (1) to state the main result. It follows from (2)–(4) that (1) is rewritten in the following form:

$$\begin{aligned} &\{\alpha(x) + \beta(x,y)\} y D_x u(x,y) \\ &+ \{a(x) + b(x,y)\} y^2 D_y u(x,y) \\ &+ u(x,y) = f(x,y), \end{aligned} \tag{9}$$

where each coefficient is holomorphic at the origin. Moreover α, β and b satisfy

$$\alpha(0) \neq 0 \tag{10}$$

$$\beta(x,0) \equiv b(x,0) \equiv 0. \tag{11}$$

In [8, 9, 10], we studied the following three cases:

Case (I) $a(x) \equiv 0$ (in [8]).

Case (II) $a(x) \equiv a \neq 0$ [non-zero constant] (in [9]).

Case (III) $\alpha(x) \equiv \alpha$ [constant] and $a(x) = ax$ [linear function] (in [10]).

In each case, we give in the next chapter additional conditions which the coefficients should satisfy in order to assure the Borel summability of the formal solution in a given direction θ.

4. Main Results

4.1 Case (I)

In this case, (9) is written as follows:

$$\{\alpha(x) + \beta(x,y)\}yD_x u(x,y) + b(x,y)y^2 D_y u(x,y) + u(x,y) = f(x,y). \tag{12}$$

Assumptions. First we state the assumption for $\alpha(x)$. Let us consider the following initial value problem:

$$\frac{dx}{d\xi} = -\alpha(x), \qquad x(0) = 0 \tag{13}$$

It is obvious that the holomorphic solution $x = \chi(\xi)$ of (13) exists uniquely on $B(r)$ for some $r > 0$. Moreover we assume the following:

(A1) The solution $x = \chi(\xi)$ of (13) exists on $S(\theta, \kappa, +\infty)$ for some $\kappa > 0$. Precisely, there exists a holomorphic function $\chi(\xi)$ on $B(r) \cup S(\theta, \kappa, +\infty)$ for some $r > 0$ and $\kappa > 0$ which satisfies (i) the image of χ is included in the domain of holomorphy of α; (ii) $\chi'(\xi) = -\alpha(\chi(\xi))$ for $\xi \in B(r) \cup S(\theta, \kappa, +\infty)$ and $\chi(0) = 0$.

Next, in order to state the assumptions for the other coefficients, we define an analytic function. Let us define the region $\Omega_{r,\theta,\kappa}$ consisting of the image of χ by

$$\Omega_{r,\theta,\kappa} = \{\chi(\xi);\ \xi \in B(r) \cup S(\theta, \kappa, +\infty)\}. \tag{14}$$

Assumption (A1) and (10) imply that $\alpha(x)$ is analytic on $\Omega_{r,\theta,\kappa}$ and that $\alpha(x) \neq 0$ for all $x \in \Omega_{r,\theta,\kappa}$.

So, let us define the function $A(x)$ on $\Omega_{r,\theta,\kappa}$ by

$$A(x) = -\int_0^x \frac{dz}{\alpha(z)}, \qquad x \in \Omega_{r,\theta,\kappa}. \tag{15}$$

Here the path of integration is the solution curve of (13). Then $A(x)$ is well defined on $\Omega_{r,\theta,\kappa}$ and it is analytic there.

Under the above preparations we give the conditions for the other coefficients. A global analytic continuation property with respect to x-variable is required:

(A2) $\beta(x,y)$, $b(x,y)$ and $f(x,y)$ can be continued analytically to $\Omega_{r,\theta,\kappa} \times \{y \in \mathbb{C};\ |y| \leq c\}$ for some $c > 0$. Moreover, they satisfy the following estimates there:

$$\sup_{x \in \Omega_{r,\theta,\kappa},\, |y| \leq c} \left| \frac{\beta(x,y)}{\alpha(x)} \right| < \infty; \tag{16}$$

$$\max_{|y| \leq c} |b(x,y)| \leq \frac{K}{\{1 + |A(x)|\}^p}, \qquad x \in \Omega_{r,\theta,\kappa}; \tag{17}$$

$$\max_{|y| \leq c} |f(x,y)| \leq C \exp[\delta |A(x)|], \qquad x \in \Omega_{r,\theta,\kappa}, \tag{18}$$

where K, C and δ are some positive constants independent of $x \in \Omega_{r,\theta,\kappa}$ and y with $|y| \leq c$. p is the constant satisfying $p > 1$.

In [8] we obtained the following theorem:

Theorem 3 ([8]) *Under assumptions* (A1) *and* (A2) *the formal solution* $u(x,y)$ *of* (12) *is Borel summable in the direction* θ.

It should be remarked that the existence of the Borel sum, which is a *local* solution, is ensured by the *global* conditions such as (A1) and (A2).

Remark 2 By applying Cauchy's integral formula, we see that (16) and (17) are equivalent to the following estimates (19) and (20), respectively. There exist some positive constants K and L such that

$$\left| \frac{1}{\alpha(x)} \frac{\partial^m \beta}{\partial y^m}(x,0) \right| \leq K L^m m!, \qquad x \in \Omega_{r,\theta,\kappa}, \quad m = 1, 2, \ldots; \tag{19}$$

$$\left| \frac{\partial^m b}{\partial y^m}(x,0) \right| \leq K L^m m! \frac{1}{\{1 + |A(x)|\}^p}, \qquad x \in \Omega_{r,\theta,\kappa}, \quad m = 1, 2, \ldots. \tag{20}$$

In the following two cases, we will give the conditions in such forms as (19) and (20).

4.2 Case (II)

In this case, (9) is written as follows:

$$\{\alpha(x)+\beta(x,y)\}yD_xu(x,y) + \{a+b(x,y)\}y^2D_yu(x,y) + u(x,y)=f(x,y). \tag{21}$$

Before giving the main result, we introduce the notation. First, we define the function $F(\tau)$ by

$$F(\tau)=\frac{1}{a}\log(1+a\tau). \tag{22}$$

Next, we define the region $\varXi_{r,\theta,\kappa}$ by

$$\varXi_{r,\theta,\kappa}=\{F(\tau);\ \tau\in B(r)\cup S(\theta,\kappa,+\infty)\}. \tag{23}$$

In order to ensure the well-definedness of $\varXi_{r,\theta,\kappa}$, we always assume

$$\theta\neq\arg\Big(-\frac{1}{a}\Big). \tag{24}$$

Assumptions. We state the assumption for $\alpha(x)$. In this case also, we consider the initial value problem (13).

(A1)′ (13) has a holomorphic solution $x=\chi(\xi)$ on $\varXi_{r,\theta,\kappa}$ for some $r>0$ and $\kappa>0$. Precisely, there exists a holomorphic function $\chi(\xi)$ on $\varXi_{r,\theta,\kappa}$ for some $r>0$ and $\kappa>0$ which satisfies (i) the image of χ is included in the domain of holomorphy of α; (ii) $\chi'(\xi)=-\alpha(\chi(\xi))$ for $\xi\in\varXi_{r,\theta,\kappa}$ and $\chi(0)=0$.

Next, we define the analytic function $A(x)$ similarly to (15) as follows. Let us define the region $\varPhi_{r,\theta,\kappa}$ consisting of the image of χ by

$$\varPhi_{r,\theta,\kappa}=\{\chi(\xi);\ \xi\in\varXi_{r,\theta,\kappa}\}. \tag{25}$$

Assumption (A1)′ and (10) imply that $\alpha(x)$ is analytic on $\varPhi_{r,\theta,\kappa}$ and that $\alpha(x)\neq0$ for all $x\in\varPhi_{r,\theta,\kappa}$. So, we define the function $A(x)$ on $\varPhi_{r,\theta,\kappa}$ by

$$A(x)=-\int_0^x\frac{dz}{\alpha(z)},\qquad x\in\varPhi_{r,\theta,\kappa}. \tag{26}$$

Here the path of integration is the solution curve of (13). Then $A(x)$ is well defined on $\varPhi_{r,\theta,\kappa}$ and it is analytic there.

Under the above preparations we give the conditions for the other coefficients. For the inhomogeneity term $f(x,y)$ we assume the following.

(A2)′ $f(x,y)$ can be continued analytically to $\varPhi_{r,\theta,\kappa}\times\{y\in\mathbb{C};\ |y|\le c\}$ for some $c>0$. Moreover, it has the following growth estimate there. There exist some positive constants C and δ such that

$$\max_{|y|\le c}|f(x,y)|\le C\exp[\delta\,|\exp\{aA(x)\}|],\quad x\in\varPhi_{r,\theta,\kappa}. \tag{27}$$

For the coefficients $\beta(x,y)$ and $b(x,y)$, we impose the following conditions.

(A3)′ $\beta(x,y)$ and $b(x,y)$ can be continued analytically to $\varPhi_{r,\theta,\kappa}\times\{y\in\mathbb{C};\ |y|\le c\}$. Moreover, there exist some positive constants K and L, which are independent of m, and $p_m<m$ such that

$$\left|\frac{1}{\alpha(x)}\frac{\partial^m\beta}{\partial y^m}(x,0)\right|\le KL^m m!|\exp\{aA(x)\}|^{p_m},\quad x\in\varPhi_{r,\theta,\kappa},\ \ m=1,2,\dots; \tag{28}$$

$$\left|\frac{\partial^m b}{\partial y^m}(x,0)\right|\le KL^m m!|\exp\{aA(x)\}|^{p_m},\quad x\in\varPhi_{r,\theta,\kappa},\ \ m=1,2,\dots. \tag{29}$$

Finally, we assume that

(A4)′ $\inf\{m-p_m;\ m=1,2,\dots\}>0$.

In [9] we obtained the following theorem:

Theorem 4 ([9]) *Under assumptions* (A1)′–(A4)′ *the formal solution* $u(x,y)$ *of* (21) *is Borel summable in the direction* θ.

4.3 Case (III)

In this case, (9) is written as follows:

$$\{\alpha+\beta(x,y)\}yD_xu(x,y) + \{ax+b(x,y)\}y^2D_yu(x,y) + u(x,y)=f(x,y) \tag{30}$$

We remark that $\alpha\neq0$ by (10).

Assumptions. First let us consider the following initial value problem:

$$\frac{d\xi}{d\tau}=\exp\Big(\frac{\alpha a}{2}\xi^2\Big),\qquad \xi(0)=0. \tag{31}$$

Then we assume the following:

(A1)′′ (31) has a holomorphic solution $\xi = \mathcal{F}(\tau)$ on the region $B(r) \cup S(\theta, \kappa, +\infty)$ for some $r > 0$ and $\kappa > 0$.

It is obvious that $\mathcal{F}(\tau)$ is unique, if it exists.

Next, let us define the region $\mathcal{X}_{r,\theta,\kappa}$ consisting of the image of $\mathcal{F}$ by

$$\mathcal{X}_{r,\theta,\kappa} = \{\mathcal{F}(\tau);\ \tau \in B(r) \cup S(\theta, \kappa, +\infty)\}, \qquad (32)$$

and let us assume the following:

(A2)′′ $\sup_{\xi \in \mathcal{X}_{r,\theta,\kappa}} \left| \exp\left(\frac{\alpha a}{2}\xi^2\right) \right| < \infty.$

Next, in order to state assumptions for coefficients, we define the region $\Psi_{r,\theta,\kappa}$ by

$$\Psi_{r,\theta,\kappa} = -\alpha \cdot \mathcal{X}_{r,\theta,\kappa} = \{-\alpha \cdot \xi;\ \xi \in \mathcal{X}_{r,\theta,\kappa}\}. \qquad (33)$$

For the inhomogeneity term $f(x, y)$ we assume the following:

(A3)′′ $f(x, y)$ can be continued analytically to $\Psi_{r,\theta,\kappa} \times \{y \in \mathbb{C};\ |y| \le c\}$ for some $c > 0$. Moreover, it has the following estimate there. There exist some positive constants C and δ such that

$$\max_{|y| \le c} |f(x, y)| \le C \exp\left[\delta \left| \mathscr{F}\left(-\frac{1}{\alpha}x\right) \right| \right], \qquad x \in \Psi_{r,\theta,\kappa}, \qquad (34)$$

where $\mathscr{F}$ is the entire function defined by

$$\mathscr{F}(\xi) = \int_0^\xi \exp\left(-\frac{\alpha a}{2}\zeta^2\right) d\zeta. \qquad (35)$$

Finally, we impose the following conditions for the coefficients $\beta(x, y)$ and $b(x, y)$:

(A4)′′ $\beta(x, y)$ and $b(x, y)$ can be continued analytically to $\Psi_{r,\theta,\kappa} \times \{y \in \mathbb{C};\ |y| \le c\}$. Moreover, there exist some positive constants K, $L > 0$ and $p > 1$ such that

$$\left| \frac{\partial^m \beta}{\partial y^m}(x, 0) \right| \le K L^m m! |E(x)|^m, \qquad x \in \Psi_{r,\theta,\kappa},\quad m = 1, 2, \dots; \qquad (36)$$

$$\left| \frac{\partial^m \beta}{\partial y^m}(x, 0) \cdot ax \right| \le \frac{K L^m m! |E(x)|^{m+1}}{\{1 + |\mathscr{F}(-(1/\alpha)x)|\}^p}, \qquad x \in \Psi_{r,\theta,\kappa},\quad m = 1, 2, \dots; \qquad (37)$$

$$\left| \frac{\partial^m b}{\partial y^m}(x, 0) \right| \le \frac{K L^m m! |E(x)|^{m+1}}{\{1 + |\mathscr{F}(-(1/\alpha)x)|\}^p}, \qquad x \in \Psi_{r,\theta,\kappa},\quad m = 1, 2, \dots, \qquad (38)$$

where $E(x)$ is the entire function defined by

$$E(x) = \exp\left(-\frac{a}{2\alpha}x^2\right). \qquad (39)$$

In [10] we obtained the following theorem:

Theorem 5 ([10]) *Under assumptions* (A1)′′–(A4)′′ *the formal solution* $u(x, y)$ *of* (30) *is Borel summable in the direction* θ.

Remark 3 If $a = 0$, then the equation (30) is a special case of the equation (12). In this case, assumptions (A1)′′ and (A2)′′ are always satisfied ($\exp\left((\alpha a/2)\xi^2\right) \equiv 1$). (34) in (A3)′′ is equivalent to (18) in (A2). Moreover, we see that (36) and (38) in (A3)′′ are equivalent to (16) and (17) in (A2), respectively (cf. Remark 2), and that (37) is always satisfied. Consequently, Theorem 5 gives one of the partial generalizations of Theorem 3.

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