Existence theorems for abstract quasi-variational inequaties

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Abstract

In this paper, I present some existence theorems for quasi-variational inequalities. Quasi-variational inequality is variational ineqality whose constraint set depends upon unknown functions, that are solutions of variational inequality. Parabolic quasi-variational inequality is represented by an evolution equation with subdifferentials.

1 Introduction

Differential equations are one of the most valuable theory to analyse various phenomena. In recent, Nonlinear analysis is developing particularly, and many brandnew informations are brought by technics of analysis. Quasi-variational inequality is also useful theory, and is studied by many mathematicians.

Let X be a real reflexive Banach space and X^* be its dual. We assume that X and X^* are strictly convex and denote by $< \cdot, \cdot >$ the duality pairing between X^* and X. Given a nonlinear operator A from X into X^*, an element $g^* \in X^*$ and a closed convex subset K of X, the variational inequality is formulated as a problem to find u in X such that

$$u \in K, \quad < Au - g^*, u - w > \leq 0 \quad {}^\forall w \in K \qquad (1)$$

Variational inequality has been studied by many mathematicians, for instance see J. L. Lions and G. Stampacchia [8], F. Browder [7], H. Brézis [5], and their references.

The concept of quasi-variational inequality was introduced by A. Bensoussan and J. L. Lions [1] in order to solve some problems in the control theory. Given an operator $A : X \to X^*$, an element $g^* \in X^*$ and a family $\{K(v); v \in X\}$ of closed convex subsets of X, the quasi-variational inequality is a problem to find u in X such that

$$u \in K(u), \quad < Au - g^*, u - w > \leq 0 \quad {}^\forall w \in K(u) \quad (2)$$

As is seen from (2), the constraint $K(u)$ for the quasi-variational inequality depends upon the unknown u, which causes one of main difficulties in the mathematical treatment of quasi-variational inequalities.

2 Existence for elliptic quasi-variational inequalities

Let X be a real Banach space and X^* be its dual space, and assume that X and X^* are strictly convex. We denote by $< \cdot, \cdot >$ the duality pairing between X^* and X, and by $|\cdot|_X$ and $|\cdot|_{X^*}$ the norms of X and X^*, respectively. For various general concepts on nonlinear multivalued operators from X into X^*, for instance, monotonicity and maximal monotonicity of operators, we refer to the monograph [2]. In this section, we mean that operators are multivalued, in general. Given a general nonlinear operator A from X into X^*, we use the notations $D(A)$, $R(A)$ and $G(A)$ to denote its domain, range and graph of A. We formulate quasi-variational inequalities for a class of nonlinear operators, which is called semimonotone, from $X \times X$ into X^*.

2.1 Existence result

Definition 2.1. An operator $\tilde{A}(\cdot, \cdot) : X \times X \to X^*$ is called *semimonotone*, if $D(\tilde{A}) = X \times X$ and the following conditions (SM1) and (SM2) are satisfied:

(SM1) For any fixed $v \in X$ the mapping $u \to \tilde{A}(v, u)$ is maximal monotone form $D(\tilde{A}(v, \cdot)) = X$ into X^*.

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1) 数学科

1) Department of Mathematics

(SM2) Let u be any element of X and $\{v_n\}$ be any sequence in X such that $v_n \to v$ weakly in X. Then, for every $u^* \in \tilde{A}(v,u)$ there exists a sequence $\{u_n^*\}$ in X such that $u_n^* \in \tilde{A}(v_n,u)$ and $u_n^* \to u^*$ in X^* as $n \to +\infty$.

Let $\tilde{A}: D(\tilde{A}) := X \times X \to X^*$ be a semimonotone operator. Then we define $A: D(A) = X \to X^*$ by putting $Au := \tilde{A}(u,u)$ for all $u \in X$, which is called the operator generated by $\tilde{A}$.

Now, for an operator A generated by semimonotone operator, any $g^* \in X^*$ and a mapping $v \to K(v)$ we consider a quasi-variational inequality (3), to find $u \in X$ and $u^* \in X^*$ such that

$$\begin{cases} u \in K(u), \quad u^* \in Au, \\ < u^* - g^*, w - u > \leq 0, \quad \forall w \in K(u). \end{cases} \tag{3}$$

Theorem 2.1. *Let $\tilde{A}: D(\tilde{A}) = X \times X \to X^*$ be a bounded semimonotone operator and A be the operator generated by $\tilde{A}$. Let K_0 be a bounded, closed and convex set in X. Suppose that to each $v \in K_0$ a non-empty, bounded, closed and convex subset $K(v)$ of K_0 is assigned, and the mapping $v \to K(v)$ satisfies the following continuity conditions* (K1) *and* (K2) *:*

(K1) *If $v_n \in K_0$, $v_n \to v$ weakly in X (as $n \to \infty$), then for each $w \in K(v)$ there is a sequence w_n in X such that $w_n \in K(v_n)$ and $w_n \to w$ (strongly) in X.*

(K2) *If $v_n \to v$ weakly in X, $w_n \in K(v_n)$ and $w_n \to w$ weakly in X, then $w \in K(v)$.*

Then, for any $g^* \in X^*$, the quasi-variational inequality (3) *has at least one solution u.*

The following theorem is a slightly general version of Theorem 2.1.

Theorem 2.2. *Let $\tilde{A}: D(\tilde{A}) = X \times X \to X^*$ be a bounded semimonotone operator and A be the operator generated by $\tilde{A}$. Suppose that to each $v \in X$ a non-empty, bounded, closed and convex subset $K(v)$ of X is assigned and there is a bounded, closed and convex subset G_0 of X such that*

$$K(v) \cap G_0 \neq \emptyset, \quad \forall v \in X,$$

and

$$\inf_{w^* \in Aw} \frac{< w^*, w - v >}{|w|_X} \to \infty \quad \text{as } |w|_X \to \infty$$

$$\text{uniformly in } v \in G_0.$$

Furthermore, the mapping $v \to K(v)$ satisfies the following condition (K'1) *and the same condition* (K2) *as in Theorem 2.1.:*

(K'1) *If $v_n \to v$ weakly in X, then for each $w \in K(v)$ there is a sequence w_n in X such that $w_n \in K(v_n)$ and $w_n \to w$ in X.*

Then, for any $g^* \in X^*$, the quasi-variational inequality (3) *has at least one solution u.*

In our proof of Theorems 2.1 and 2.2, we use some results on nonlinear operators of monotone type. For detailed proof, see [11], and we show applications for elliptic quasi-variational inequalities in [14].

3 Existence for parabolic quasi-variational inequalities

For positive numbers δ_0, T, we are given sets

$$V(-\delta_0, t), \ 0 \leq t \leq T,$$

of functions from $(-\delta_0, t)$ into a real Hilbert space H and a family $\{\varphi^s(v;\cdot)\}_{0 \leq s \leq t}$ of proper, lower semicontinuous, convex functions $\varphi^s(v;\cdot)$ with parameters $s \in [0,t]$ and $v \in V(-\delta_0,t)$; here $\varphi^s(v;\cdot)$ continuously depends upon $v \in V(-\delta_0,t)$ in a certain nonlocal way. We consider a nonlinear evolution equation of the form:

$$u'(t) + \partial\varphi^t(u; u(t)) \ni f(t), \quad 0 < t < T, \quad \text{in } H, \tag{4}$$

subject to the initial condition

$$u(t) = u_0(t), \quad -\delta_0 \leq t \leq 0, \quad \text{in } H, \tag{5}$$

where $\partial\varphi^t(u;\cdot)$ is the subdifferential of convex function $\varphi^t(u;\cdot)$ on H, $u' = \frac{du}{dt}$ and $u_0 : [-\delta_0, 0] \to H$ and $f : (0,T) \to H$ are prescribed as the initial and forcing functions, respectively. This is a sort of functional differential equations generated by subdifferentials of $\varphi^t(v;\cdot)$ with a nonlocal dependence upon v. The objective of this section is to specify a class of convex functions $\{\varphi^s(v;\cdot)\}_{0 \leq s \leq t}$ as well as its nonlocal dependence upon $v \in V(-\delta_0,t)$ in order that Cauchy problem $\{(4),(5)\}$ admits at least one local or global in time solution u.

In general, for a given real Banach space X we denote by $|\cdot|_X$ the norm in X. Throughtout this section, let H be a real Hilbert space with inner product

$(\cdot,\cdot)_H$ and norm $|\cdot|_H$. Given a proper, lower semi-continuous (l.s.c.) and convex function $\psi(\cdot)$ on H we use the usual notation:

- $D(\psi) := \{z \in H; \psi(z) < \infty\}$ (so called effective domain).
- $\partial\psi$ is the subdifferential of ψ, which is a (multivalued) mapping in H and defined by

$$z^* \in \partial\psi(z) \Longleftrightarrow (z^*, v - z)_H \le \psi(v) - \psi(z), \quad \forall v \in H$$

with domain

$$D(\partial\psi) := \{z \in H; \partial\psi(z) \neq \emptyset\} (\subset D(\psi)).$$

There is an important concept of convergence for convex functions, which was introduced by Mosco [6] in order to characterize the convergence of solutions to variational inequalities. Let $\{\psi_n\}$ be a sequence of proper l.s.c. and convex functions on H. Then it is said that ψ_n converges to a proper, l.s.c. and convex function ψ on H in the sense of Mosco, if the following two conditions (M1) and (M2) are fulfilled:

(M1) $\liminf_{n\to\infty} \psi_n(z) \ge \psi(z)$ for every $z \in H$.

(M2) For each $z \in D(\psi)$ there is a sequence $\{z_n\}$ in H such that $z_n \to z$ in H and $\psi_n(z_n) \to \psi(z)$ as $n \to \infty$.

We refer various basic properties about convex functions to monographs [3, 4, 10].

3.1 Local existence result

In order to formulate functions $\varphi^t(v;\cdot)$ precisely we introduce a time-independent, non-negative, proper, l.s.c. and convex function $\varphi_0(\cdot)$ on H such that the set $\{z \in H; |z|_H \le r,\ \varphi_0(z) \le r\}$ is compact in H for each $r \ge 0$.

Let δ_0 be a fixed positive number and $T > 0$ be a finite time. For each $t \in [0,T]$ we define a closed convex subset $\mathcal{V}(-\delta_0,t)$ of $W^{1,2}(-\delta_0,t;H)$ by

$$\mathcal{V}(-\delta_0,t) := \{v; V_{[-\delta_0,t]}(v) < \infty\} \tag{6}$$

with

$$V_{[-\delta_0,t]}(v) := \sup_{-\delta_0 \le s \le t} \varphi_0(v(s)) + |v(0)|_H^2 + |v'|^2_{L^2(-\delta_0,t;H)} \tag{7}$$

where $v'(t) = \frac{dv(t)}{dt}$.

Now, to each $v \in \mathcal{V}(-\delta_0,t)$ a family $\{\varphi^s(v;\cdot)\}_{0\le s\le t}$ of functions $\varphi^s(v;\cdot)$ on H is assigned such that

(Φ1) $\varphi^s(v;z)$ is proper, l.s.c., non-negative and convex in $z \in H$, and it is determined by $s \in [0,t]$ and v on $[-\delta_0,s]$; namely, for $v_1,\ v_2 \in \mathcal{V}(-\delta_0,t)$, we have $\varphi^s(v_1,\cdot) \equiv \varphi^s(v_2,\cdot)$ on H whenever $v_1 \equiv v_2$ on $[-\delta_0,s]$;

(Φ2) $\varphi^s(v;z) \ge \varphi_0(z),\ \forall v \in \mathcal{V}(-\delta_0,t),$ $0 \le \forall s \le \forall t \le T$;

(Φ3) If $0 \le s_n \le t \le T,\ v_n \in \mathcal{V}(-\delta_0,t)$, $\sup_{n\in\mathbf{N}} V_{[-\delta_0,t]}(v_n) < \infty$, $s_n \to s$ and $v_n \to v$ in $C([-\delta_0,t];H)$, then $\varphi^{s_n}(v_n;\cdot) \to \varphi^s(v;\cdot)$ on H in the sense of Mosco.

We give the definition of solutions for evolution equation (4).

Definition 3.1. Let $u_0 \in C([-\delta_0,0];H)$ and $f \in L^2(0,T;H)$. Then we say that u is a solution of the following Cauchy problem $CP(u_0,f)$

$$\begin{cases} u'(t) + \partial\varphi^t(u;u(t)) \ni f(t),\ 0<t<T, \\ u = u_0 \quad \text{on } [-\delta_0,0] \end{cases}$$

on $[0,T]$, if u satisfies that $u \in C([-\delta_0,T];H)$, $u = u_0$ on $[-\delta_0,0]$, $u \in W^{1,2}(\delta,T;H)$ for every (small) $\delta > 0$, $\varphi^{(\cdot)}(u;u(\cdot)) \in L^1(0,T)$ and $f(t) - u'(t) \in \partial\varphi^t(u;u(t))$ for a.e. $t \in (0,T)$.

We introduce the following function spaces: given any function u_0 in $\mathcal{V}(-\delta_0,0)$, $0 < R < \infty$ and $t \in [0,T]$, we put

$$\mathcal{V}(u_0;-\delta_0,t) := \{v \in \mathcal{V}(-\delta_0,t); v = u_0 \text{ on } [-\delta_0,0]\},$$

and

$$\mathcal{V}_R(u_0;-\delta_0,t) (\subset \mathcal{V}(u_0;-\delta_0,t)) := \left\{ v \;\middle|\; \sup_{0\le s\le t} \left\{\varphi_0(v(s)) + |v'|^2_{L^2(0,s;H)}\right\} \le R \right\}.$$

Theorem 3.1. *Let $0 < T < \infty$ and $u_0 \in \mathcal{V}(-\delta_0,0)$ with $\varphi^0(u_0;u_0(0)) < \infty$. Assume that there are positive numbers $T_0 \le T$ and $R > \varphi^0(u_0;u_0(0))$, a family $\{M_r\}_{0\le r<\infty}$ of positive numbers M_r and a set $\{\{\varphi^t(v;\cdot)\}; v \in \mathcal{V}_R(u_0;-\delta_0,T_0)\}$ of families $\{\varphi^t(v;\cdot)\}_{0\le t\le T_0}$ of convex functions satisfying the following condition (*):*

(*) There are two families

$$\{a_r^v; v \in \mathcal{V}_R(u_0; -\delta_0, T_0), 0 \le r < \infty\}$$

of non-negative functions in $L^2(0, T_0)$ *and* $\{b_r^v; v \in \mathcal{V}_R(u_0; -\delta_0, T_0), 0 \le r < \infty\}$ *of non-negative functions in* $L^1(0, T_0)$ *such that*

(H1) $|a_r^v|_{L^2(0,T_0)} \le M_r$ *and* $|b_r^v|_{L^1(0,T_0)} \le M_r$ *for all* $r > 0$ *and all* $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$, *and* $\{\varphi^t(v; \cdot)\} \in G(\{a_r^v\}, \{b_r^v\})$ *for all* $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$*;*

(H2) for each finite $r > 0$ *and* $\varepsilon > 0$ *there is a positive number* $\delta_{r\varepsilon} > 0$ *such that*

$$\int_0^{\delta_{r\varepsilon}} (a_r^v(\tau)^2 + b_r^v(\tau))d\tau < \varepsilon,$$
$$\forall v \in \mathcal{V}_R(u_0; -\delta_0, T_0).$$

Then, for each $f \in L^2(0, T_0; H)$, *problem* $CP(u_0, f)$ *has at least one solution* u *on an interval* $[0, T']$ *with* $0 < T' \le T_0$ *such that* $u \in \mathcal{V}(-\delta_0, T'; H)$ *and* $\sup_{0 \le t \le T'} \varphi^t(u; u(t)) < \infty$.

(Sketch of the proof.)

For fixed $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$ we can get unique solution u of

$$\begin{cases} u_t + \partial\varphi^t(v; u(t)) \ni f(t) \quad \text{a.e. } t \in [0, T_0], \\ u(0) = u_0. \end{cases} \tag{8}$$

under our assumptions. With using fixed point theorem, we can see that there exists fixed point $v = u$. This is a time local solution u of $CP(u_0, f)$.

If you need detailed proof, see [12]. It should be noticed that the family of proper, l.s.c., convex functions $G(\{a_r^v\}, \{b_r^v\}$ is essential to solve (8). To see the definition of this family, see [9].

3.2 Global existence result

Let φ_0 be the same as in the previous section as well as $\delta_0 > 0$ and $T > 0$. In this section, we consider a closed convex subset $\tilde{\mathcal{V}}(-\delta_0, t)$ of $L^2(-\delta_0, t; H)$ for each $t \in [0, T]$, as is defined below, in place of $\mathcal{V}(-\delta_0, t)$.

For each $t \in [0, T]$ we define

$$\tilde{\mathcal{V}}(-\delta_0, t) := \{v; \tilde{V}_{[-\delta_0,t]}(v) < \infty\}, \tag{9}$$

where

$$\tilde{V}_{[-\delta_0,t]}(v) := |v|^2_{L^\infty(-\delta_0,t;H)} + \int_{-\delta_0}^t \varphi_0(v(s))ds. \tag{10}$$

Now, we suppose that to each $v \in \tilde{\mathcal{V}}(-\delta_0, t)$ a family $\{\varphi^s(v; \cdot)\}_{0 \le s \le t}$ of functions $\varphi^s(v; \cdot)$ on H is assigned such that

($\tilde{\Phi}1$) $\varphi^s(v; z)$ is proper, l.s.c., non-negative and convex in $z \in H$, and it is determined by $s \in [0, t]$ and v on $[-\delta_0, s]$; namely, for v_1, $v_2 \in \tilde{\mathcal{V}}(-\delta_0, t)$, we have $\varphi^s(v_1, \cdot) \equiv \varphi^s(v_2, \cdot)$ on H whenever $v_1 = v_2$ a.e. on $(-\delta_0, s)$;

($\tilde{\Phi}2$) $\varphi^s(v; z) \ge \varphi_0(z)$, $\forall v \in \tilde{\mathcal{V}}(-\delta_0, t)$, $0 \le \forall s \le \forall t \le T$;

($\tilde{\Phi}3$) If $0 \le s_n \le t \le T$, $v_n \in \tilde{\mathcal{V}}(-\delta_0, t)$, $\sup_{n \in \mathbf{N}} \tilde{V}_{[-\delta_0,t]}(v_n) < \infty$, $s_n \to s$ and $v_n \to v$ in $L^2(-\delta_0, t; H)$, then $\varphi^{s_n}(v_n; \cdot) \to \varphi^s(v; \cdot)$ on H in the sense of Mosco.

Next, we define a function space $\tilde{\mathcal{V}}_M(-\delta_0, t)$ for each $M > 0$ and $t \in [0, T]$ by

$$\tilde{\mathcal{V}}_M(-\delta_0, t) := \{v \in \tilde{\mathcal{V}}(-\delta_0, t);\ \tilde{V}_{[-\delta_0,t]}(v) \le M\}.$$

In order to show the existence of a solution of $CP(u_0, f)$ on the whole interval $[0, T]$ we relax assumptions (H1) and (H2) as follows: For each $M > 0$ there is a family $\{M_r\}_{0 \le r < \infty}$ of positive numbers M_r and a set $\{\{\varphi^t(v; \cdot)\}; v \in \tilde{\mathcal{V}}_M(-\delta_0, T)\}$ of families $\{\varphi^t(v; \cdot)\}_{0 \le t \le T}$ of convex functions satisfying the following condition (**):

(**) There are two families

$$\{a_r^v; v \in \tilde{\mathcal{V}}_M(-\delta_0, T), 0 \le r < \infty\}$$

of non-negative functions in $L^2(0, T)$ and $\{b_r^v; v \in \tilde{\mathcal{V}}_M(-\delta_0, T), 0 \le r < \infty\}$ of non-negative functions in $L^1(0, T)$ such that

($\tilde{H}1$) $|a_r^v|_{L^2(0,T)} \le M_r$ and $|b_r^v|_{L^1(0,T)} \le M_r$ for all $r > 0$ and all $v \in \tilde{\mathcal{V}}_M(-\delta_0, T)$, and $\{\varphi^t(v; \cdot)\} \in G(\{a_r^v\}, \{b_r^v\})$ for all $v \in \tilde{\mathcal{V}}_M(-\delta_0, T)$;

($\tilde{H}2$) for each finite $r > 0$ and $\varepsilon > 0$ there is a positive number $\delta_{r\varepsilon} > 0$ such that

$$\int_t^{t+\delta_{r\varepsilon}} (a_r^v(\tau)^2 + b_r^v(\tau))d\tau < \varepsilon,$$
$$\forall t \in [0, T - \delta_{r\varepsilon}],\ \forall v \in \tilde{\mathcal{V}}_M(-\delta_0, T).$$

It should be noted that these conditions are independent of initial data. Moreover we require the following assumption ($\tilde{H}3$):

$(\tilde{H}3)$ there are a positive number R_0 and a family $\{h_v\} := \{h_v; v \in \tilde{\mathcal{V}}(-\delta_0, T)\}$ of functions in $W^{1,2}(0,T;H)$ such that

$$|h_v|_{W^{1,2}(0,T;H)} \le R_0,$$
$$\int_0^T \varphi^t(v; h_v(t))dt \le R_0, \quad \forall v \in \tilde{\mathcal{V}}(-\delta_0, T).$$

We first show the existence of a solution $CP(u_0, f)$ on the whole interval $[0,T]$ for good initial values u_0.

Theorem 3.2. *Suppose that $(\tilde{H}1)$ and $(\tilde{H}2)$ hold for every $M > 0$ as well as $(\tilde{H}3)$. Let $u_0 \in \mathcal{V}(-\delta_0, 0)$ with $\varphi^0(u_0; u_0(0)) < \infty$ and f be any function in $L^2(0,T;H)$. Then $CP(u_0, f)$ has at least one solution u on $[0,T]$ such that*

$$u \in W^{1,2}(0,T;H), \qquad \sup_{0\le t\le T} \varphi^t(u; u(t)) < \infty.$$

Before ending this section, I show the existence of a solution of $CP(u_0, f)$ for a little bit more general class of initial data.

Theorem 3.3. *Suppose that $(\tilde{\Phi}1)$, $(\tilde{\Phi}2)$ and $(\tilde{\Phi}3)$ hold and that $(\tilde{H}1)$ and $(\tilde{H}2)$ hold for every $M > 0$ as well as $(\tilde{H}3)$. Let $u_0 \in \tilde{\mathcal{V}}(-\delta_0, 0) \cap C([-\delta_0, 0]; H)$ such that there is a sequence $\{u_{0n}\}$ in $\mathcal{V}(-\delta, 0)$ with $\varphi^0(u_{0n}; u_{0n}(0)) < \infty$ satisfying that*

$$\sup_{n\in\mathbf{N}} \tilde{V}_{[-\delta_0,0]}(u_{n0}) < \infty, \;\; u_{n0} \to u_0 \;\text{ in } C([-\delta_0, 0]; H).$$

Then $CP(u_0, f)$ has at least one solution u on $[0,T]$ such that

$$\begin{cases} u \in C([0,T];H), \quad \sqrt{t}u' \in L^2(0,T;H), \\ \sup_{0<t\le T} t\varphi^t(u; u(t)) < \infty. \end{cases}$$

Theorem 3.2 and 3.3 are proved in [12].

4 Applications

In this section, we consider a nonlinear system $\mathrm{SP}(u_0, \theta_0; f)$ of the following form:

$$\begin{cases} u_t - \nu\Delta u + g(\theta, u) + \partial I_{K(E)}(u) \ni 0 & \text{in } Q \\ E = \mathcal{E}(\theta, u) & \text{in } Q \\ \theta_t - \kappa\Delta\theta + h(\theta, u) = f & \text{in } Q \\ \dfrac{\partial u}{\partial n} = 0, \quad \theta = 0 & \text{on } \Sigma \\ u = u_0, \quad \theta = \theta_0 & \text{in } Q_0 \end{cases}$$

where Ω is a smooth bounded domain of $\mathbf{R}^N$, $\Gamma = \partial\Omega$, $Q := (0,T)\times\Omega$, $\Sigma := (0,T)\times\Gamma$, $Q_0 := [-1,0]\times\Omega$, $0 < T < \infty$, $(\cdot)_t := \frac{\partial(\cdot)}{\partial t}$, and $\frac{\partial}{\partial n}$ denotes the outward normal derivative on Γ; κ and ν are positive constants and f is a function given on Q; w_0 and θ_0 are prescribed on $\Omega \times [-1, 0]$ as initial conditions; $g(\cdot,\cdot)$ is a vector field from $\mathbf{R}\times\mathbf{R}^2$ into $\mathbf{R}^2$, $u := (u^{(1)}, u^{(2)}) \in \mathbf{R}^2$, $h(\cdot,\cdot)$ is a function on $\mathbf{R}\times\mathbf{R}^2$, $K(E)$ is a non-empty compact convex subset of $\mathbf{R}^2$ for each $E \in \mathbf{R}$ and $\partial I_{K(E)}$ is the subdifferential of the indicator function $I_{K(E)}$ of $K(E)$ in $\mathbf{R}^2$; $\mathcal{E}$ is an operator from a function space $(\subset L^2(Q)\times L^2(Q)^2)$ into the space of smooth functions on Q.

For instance, in the biological context, consider the coexistence or competition models of two species of bacteria A and B. Now, let us pay attention to the temperature field θ as the most important parameter which controls the power of activation of bacteria A and B. Let $u := (u^{(1)}, u^{(2)})$ be the densities (or the parameter indicating the activation) of bacteria A and B and that their dynamics are governed by a reaction-diffusion equation. E is described by $E = \mathcal{E}(\theta, u)$ via a non-local smoothing operator $\mathcal{E}$, for instance, defined by

$$[\mathcal{E}(\theta,u)](t,x) := \int_{-1}^t \int_\Omega \rho(x-y, t-s, ; \theta(s,y), u(s,y))dyds, \quad \forall (t,x) \in Q,$$

where $\rho(\cdot,\cdot;\cdot,\cdot)$ is a smooth function on $\mathbf{R}^N\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}^2$ with $\rho(x,s;\theta,u) = 0$ if $s \ge 1$.

Now we put some assumptions for the system $\mathrm{SP}(u_0, \theta_0; f)$.

(A1) $g(\cdot,\cdot) = (g_1(\cdot,\cdot), g_2(\cdot,\cdot))$ is a globally bounded, Lipschitz continuous vector field from $\mathbf{R}\times\mathbf{R}^2$ into $\mathbf{R}^2$.

(A2) $h(\cdot,\cdot)$ is a globally bounded and Lipschitz continuous function from $\mathbf{R}\times\mathbf{R}^2$ into $\mathbf{R}$.

In order to mention precisely the assumptions on $K(E)$ we need the space $C^1(\mathbf{R}^2, \mathbf{R}^2)$ of all C^1-transformations from $\mathbf{R}^2$ into itself, equipped with the usual metric, and $\mathcal{D}^1(\mathbf{R}^2)$ of all C^1-diffeomorphisms in $\mathbf{R}^2$, which is a subset of $C^1(\mathbf{R}^2, \mathbf{R}^2)$.

(A3) $K(\cdot)$ is a set-valued mapping from $\mathbf{R}$ into $2^{\mathbf{R}^2}$ such that $K(E)$ is a non-empty, compact and convex subset of $\mathbf{R}^2$ for each $E \in \mathbf{R}$. Suppose that

(a1) $\bigcup_{E\in\mathbf{R}} K(E)$ is bounded in $\mathbf{R}^2$, and $\bigcap_{E\in\mathbf{R}} K(E) \neq \emptyset$.

(a2) for each $E \in \mathbf{R}$ there is $X_E(\cdot) \in \mathcal{D}^1(\mathbf{R}^2)$ such that $X_E(K(0)) = K(E)$,

(a3) the mappings $E \to X_E(\cdot)$ and $E \to \frac{d}{dE}X_E(\cdot)$ are continuous from $\mathbf{R}$ into $C^1(\mathbf{R}^2, \mathbf{R}^2)$.

(A4) $\mathcal{E}$ is an operator from $D_{\mathcal{E}}$ which is defined by

$$\bigcup_{t\in[0,T]} L^\infty(-1,t;L^2(\Omega)) \times L^\infty(-1,t;L^2(\Omega)^2)$$

into $C^1(\overline{\Omega})$, and enjoy the following conditions (b1), (b2) and (b3):

(b1) For each $\theta \in L^\infty(-1,t;L^2(\Omega))$ and $w \in L^\infty(-1,t;L^2(\Omega)^2)$ with $t \in [0,T]$ we put

$$E(s,x) := \mathcal{E}(\theta_{|[-1,s]}, u_{|[-1,s]})(x), \quad \forall (s,x) \in [0,t] \times \overline{\Omega}$$

where $\theta_{|[-1,s]}$ and $w_{|[-1,s]}$ are respectively the restrictions of θ and u on $[-1,s] \times \Omega$. Suppose that $s \to E(s,\cdot)$ is a Lipschitz continuous function from $[0,t]$ into $C^1(\overline{\Omega})$ with $\frac{d}{ds}E \in L^\infty(0,t;C^1(\overline{\Omega}))$; hence

$$E(s_1,\cdot) - E(s_2,\cdot) = \int_{s_2}^{s_1} \frac{d}{d\tau}E(\tau,\cdot)d\tau \ \text{ in } C^1(\overline{\Omega}), \quad \forall s_1,\ s_2 \in [0,t].$$

(b2) Suppose that $\mathcal{E}$ is continuous in the following sense: if $\{\theta_n\}$ and $\{w_n\}$ are bounded in $L^\infty(-1,t;L^2(\Omega))$ and $L^\infty(-1,t;L^2(\Omega)^2)$ for each $t \in [0,T]$, respectively, and if $\theta_n \to \theta$ in $L^2(-1,t;L^2(\Omega))$ and $u_n \to u$ in $L^2(-1,t;L^2(\Omega)^2)$, then

$$E_n(s,x) := \mathcal{E}(\theta_{n|[-1,s]}, u_{n|[-1,s]})(x)$$
$$\to E(s,x) := \mathcal{E}(\theta_{|[-1,s]}, u_{|[-1,s]})(x)$$
$$\text{in} \quad C([0,t];C^1(\overline{\Omega}))$$

(b3) If θ and w vary in bounded subsets of $L^\infty(-1,t;L^2(\Omega))$ and $L^\infty(-1,t;L^2(\Omega)^2)$ for each $t \in [0,T]$, respectively, then $E(s,x) := \mathcal{E}(\theta_{|[-1,s]}, u_{|[-1,s]})(x)$ and its derivative $\frac{d}{dt}E$ vary in bounded subsets of $C([0,t];C^1(\overline{\Omega}))$ and $L^\infty(0,t;C^1(\overline{\Omega}))$, respectively.

For some typical examples of $K(E)$ and $\mathcal{E}(\theta,u)$, see [15] and [17].

Definition 4.1 Let $f \in L^2(0,T;L^2(\Omega))$, $\theta_0 \in C([-1,0];L^2(\Omega))$ and $u_0 := (u_0^{(1)}, u_0^{(2)})$ belongs to $C([-1,0];L^2(\Omega)^2)$. Then a set of functions $\{\theta, u := (u^{(1)}, u^{(2)})\}$ is called a solution of SP$(u_0,\theta_0;f)$ on $[0,T]$, if the following conditions (s1)-(s4) are fulfilled:

(s1) $\theta \in C([-1,T];L^2(\Omega)) \cap W^{1,2}_{loc}((0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^2_{loc}((-1,0];H^2(\Omega))$, $\theta = \theta_0$ on $[-1,0]$,
$u \in C([-1,T];L^2(\Omega)^2) \cap W^{1,2}_{loc}((0,T];L^2(\Omega)^2) \cap L^2(0,T;H^1(\Omega)^2)$ and $u = u_0$ on $[-1,0]$.

(s2) For a.e. $t \in (0,T)$, the following equation holds:
$\theta'(t) - \kappa\Delta_D\theta(t) + h(\theta(t),u(t)) = f(t)$ in H,
where Δ_D stands for the Laplacian with homogeneous Dirichlet boundary condition.

(s3) $u(t,x) \in K(E(t,x))$ for a.e. $(t,x) \in Q$, where $E(t,x) := \mathcal{E}(\theta_{|[-1,t]}, u_{|[-1,t]})(x)$ for $(t,x) \in \overline{Q}$.

(s4) For a.e. $t \in (0,T)$, the following quasi-variational inequality holds:

$$\int_\Omega (u'(t,x) + g(\theta(t,x),u(t,x))) \cdot (u(t,x) - z(x))dx + \nu \int_\Omega \nabla u(t,x) \cdot \nabla(u(t,x) - z(x))dx \leq 0,$$
$$\forall z \in H^1(\Omega)^2 \text{ with } z(x) \in K(E(t,x)) \text{ for a.e. } x \in \Omega,$$

Theorem 4.1 *Suppose that assumptions (A1)-(A4) are fulfilled. Let $f \in L^2(0,T;L^2(\Omega))$ and $\theta_0 \in C([-1,0];L^2(\Omega))$ as well as $u_0 := (u_0^{(1)}, u_0^{(2)})$ belongs to $C([-1,0];L^2(\Omega)^2)$. Suppose that there are sequences $\{\theta_{0n}\}$ in $C([-1,0];H^1_0(\Omega))$ and $\{u_{0n} := (u_{0n}^{(1)}, u_{0n}^{(2)})\}$ in $C([-1,0];H^1(\Omega)^2)$ such that*

$$\begin{cases} \theta_{0n} \to \theta_0 \ \textit{in}\ C([-1,0];L^2(\Omega)), \\ u_{0n} \to u_0 \ \textit{in}\ C([-1,0];L^2(\Omega)^2), \end{cases}$$

$u_{0n}(0,x) \in K(E_n(x))$ for a.e. $x \in \Omega$, $\forall n = 1, 2, \cdots$,

where $E_n(x) := \mathcal{E}(\theta_{0n}, u_{0n})(x)$ for all $x \in \overline{\Omega}$. Then, SP($u_0, \theta_0; f$) admits at least one solution $\{\theta, u := (u^{(1)}, u^{(2)})\}$ on $[0,T]$ in the sense of Definition 4.1 such that

$$\sqrt{t}\theta' \in L^2(0,T;L^2(\Omega)), \quad t|\nabla\theta|^2_{L^2(\Omega)} \in L^\infty(0,T),$$

and

$$\sqrt{t}u' \in L^2(0,T;L^2(\Omega)^2), \quad t|\nabla u|^2_{L^2(\Omega)^2} \in L^\infty(0,T),$$

where $|\nabla u|^2_{L^2(\Omega)^2} := |\nabla u^{(1)}|^2_{L^2(\Omega)} + |\nabla u^{(2)}|^2_{L^2(\Omega)}$.

For the detailed proof of this theorem, see [15].

We have to introduce a regularized problem for SP($u_0, \theta_0; f$) to analyse SP($u_0, \theta_0; f$). For every positive constant ε, we define a problem SP$_\varepsilon(u_0, \theta_0; f)$ as:

$$\begin{cases} u_t - \nu\Delta u + g(\theta, u) + \partial I^\varepsilon_{K(E)}(u) = 0 & \text{in } Q \\ E = \mathcal{E}(\theta, u) & \text{in } Q \\ \theta_t - \kappa\Delta\theta + h(\theta, u) = f & \text{in } Q \\ \dfrac{\partial u}{\partial n} = 0, \quad \theta = 0 & \text{on } \Sigma \\ u = u_0, \quad \theta = \theta_0 & \text{in } Q_0 \end{cases}$$

where $I^\varepsilon_{K(E)}$ is the Moreau-Yosida regularization of $I_{K(E)}$, namely

$$I^\varepsilon_{K(E)}(u) = \inf_{w \in K(E)} \left(\frac{1}{2\varepsilon}|w-u|^2 \right) \quad \forall u \in \mathbf{R}^2,$$

and $\partial I^\varepsilon_{K(E)}$ is the Yosida approximation of the subdifferential $\partial I_{K(E)}$ of $I_{K(E)}$, namely

$$\partial I^\varepsilon_{K(E)} = \frac{I - P_{K(E)}}{\varepsilon},$$

where $P_{K(E)}$ is the projection from $\mathbf{R}^2$ onto $K(E)$.

As was proved in section 6 of [15], SP$_\varepsilon(u_0, \theta_0; f)$ has a unique solution $\{\theta_\varepsilon, u_\varepsilon = (u_\varepsilon^{(1)}, u_\varepsilon^{(2)})\}$ under (A1)-(A4) with the following additional assumptions (A5) and (A6).

(A5) $|P_{K(E_1)}(v) - P_{K(E_2)}(v)| \le C_P|E_1 - E_2|$, $\forall E_1$, $E_2 \in \mathbf{R}, \forall v \in B_0$, where B_0 is a closed ball around the origin in $\mathbf{R}^2$ with $\cup_{E\in\mathbf{R}} K(E) \subset B_0$ and C_P is a positive constant.

(A6) The environment index $E(x,t) := \mathcal{E}(\theta_{|[-1,t]})(x)$ depends only on temperature θ and is given by an integral operator of the form

$$\mathcal{E}(\theta_{|[-1,t]})(x) := \int_{-1}^{t} \int_\Omega \rho(x-y, t-s; \theta(s,y))dyds, \quad \forall (t,x) \in \overline{Q},$$

where $\rho(\cdot,\cdot;\cdot)$ is smooth on $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}$ and $\rho(y,s;r) = 0$ if $s \ge 1$.

The next theorem ensures that all the solutions of SP($u_0, \theta_0; f$) are approximated by regular problems SP$_\varepsilon(u_0, \theta_0; f_\varepsilon)$ of the above type, although problem SP($u_0, \theta_0; f$) has multiple solutions in general.

In the rest of this paper, in order to avoid some irrelevant arguments we assume that the initial data $\{\theta_0, u_0\}$ satisfy $\theta_0 \in C([-1,0]; H^1_0(\Omega))$, $u_0 \in C([-1,0]; H^1(\Omega)^2)$ and $u_0 \in K(E_0)$ a.e. in Ω with

$$E_0(x) := \int_{-1}^{0} \int_\Omega \rho(x-y, -s, \theta_0(s,y))dyds, \quad x \in \Omega.$$

Theorem 4.2 *Let $\{\theta, u\}$ be any solution of problem SP($u_0, \theta_0; f$), and let $\{\varepsilon_n\}$ be any sequence of positive numbers with $\varepsilon_n \to 0$ (as $n \to \infty$). Then there exists a sequence $\{\theta_n, u_n\}$ of solutions of $SP_{\varepsilon_n}(u_0, \theta_0; f_n)$ such that $\theta_n \to \theta$ in $C([0,T]; L^2(\Omega))$, weakly* in $L^\infty(0,T; H^1_0(\Omega))$ and weakly in $W^{1,2}(0,T; L^2(\Omega))$ as well as $u_n \to u$ in $C([0,T]; L^2(\Omega)^2)$, and weakly* in $L^\infty(0,T; H^1_0(\Omega)^2)$ and weakly in $W^{1,2}(0,T; L^2(\Omega)^2)$, and $f_n - f \to 0$ in $L^\infty(0,T; L^2(\Omega))$.*

For the detailed proof of Theorem 4.1 and 4.2, see [15].

We proved existence theorems of optimal control problem for SP($u_0, \theta_0; f$) and its approximated problems in [16]. Moreover, we show existence of solutions for several time-discrete problem of SP($u_0, \theta_0; f$). Results in [16] give us numerical scheme to get one of the solutions of SP($u_0, \theta_0; f$) numerically. Analysing time-discrete problem is very important from numerical point of view.

If you need to see other examples, you can see concrete application of quasi-variational inequality in [13, 17].

References

[1] A. Bensoussan and J.L. Lions: Nouvelle formulation de problémes de controle impulsionnel et

applications, C. R. Acad. Sci. Paris Sér. A, 276 (1973), 1189–1192.

[2] C.Baiocchi and A.Capelo: Variational and Quasivariational Inequalities, John Wiley and Sons, Chichester-New York-Brisbane-Toronto-Singapore, 1984.

[3] A. Attouch: Variational Convergence for Functions and Operators, Pitman Advanced Publishing Program, Pitman, Boston, 1984.

[4] H. Brézis: Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland, Amsterdam, 1973.

[5] H.Brézis: Équations et inéaires dans les espaces vectoriels en dualité, Ann. Inst. Fourier, Grenoble, 18 (1968), 115–175.

[6] U. Mosco: Convergence of convex sets and of solutions of variational inequalities, Advances in Math., 3 (1969), 510–585.

[7] F.E. Browder: Nonlinear monotone operators and convex sets in Banach spaces, Bull. Amer. Math. Soc., 71 (1965), 780–785.

[8] J.L. Lions and G. Stampacchia: Variational inequalities, Comm. Pure Appl. Math., 20 (1967), 493-519.

[9] N. Kenmochi: Solvability of nonlinear evolution equations with time-dependent constraints and applications, Bull. Fac. Education, Chiba Univ., 30 (1981), 1-87.

[10] N. Kenmochi: Monotonicity and Compactness Methods for Nonlinear Variational Inequalities, HANDBOOK OF DIFFERENTIAL EQUATIONS, Stationary Partial Differential Equations, volume 4, Elsevier B.V., (2008), 203–298.

[11] R. Kano, N. Kenmochi, Y. Murase: An existence theorem for elliptic quasi-variational inequalities in Banach spaces. Recent Advance in Nonlinear Analysis: Proceedings of the International Conference on Nonlinear Analysis (2008), 149–170.

[12] R. Kano, N. Kenmochi, Y. Murase: Nonlinear evolution equations generated by subdifferentials with nonlocal constraints, Banach Center Publications, Inst. Math. Polish Acad. Sci., 86 (2009), 175–194.

[13] R. Kano, N. Kenmochi, Y. Murase: Parabolic quasi-variational inequalities with nonlocal constraints, Adv. Math. Sci. Appl., 19 No.2 (2009), 565–583.

[14] R. Kano, N. Kenmochi, Y. Murase: Elliptic quasi-variational inequalities and applications, Discrete Contin. Dyn. Syst. 2009, Dynamical Systems, Differential Equations and Applications. 7th AIMS Conference, suppl., 583–591.

[15] A. Kadoya, N. Kenmochi, Y. Murase: A class of nonlinear parabolic systems with environmental constraints, Adv. Math. Sci. Appl., 20 No.1 (2010), 281–313.

[16] A. Kadoya, N. Kenmochi, Y. Murase: Optimal control problems for quasi-variational inequalities and its numerical approximation. (To Be Published)

[17] Y. Murase and A. Ito: Mathematical modeling and analysis for the brewing process of Japanese sake, Kōkyūroku of RIMS, Kyoto Univ., 1746 (2011), 141–161. (Japanese)

(原稿受理日　平成 23 年 9 月 30 日)